The Cauchy–Goursat Theorem

Rectangle Bisection, the Improved Goursat Theorem

Lecture Notes

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Édouard Goursat (1858–1936)

Theorem (Cauchy–Goursat)

Let R be a rectangle, $f \in \mathcal{A}(R)$. Then $\oint_{\partial R} f(z)\,dz = 0$.

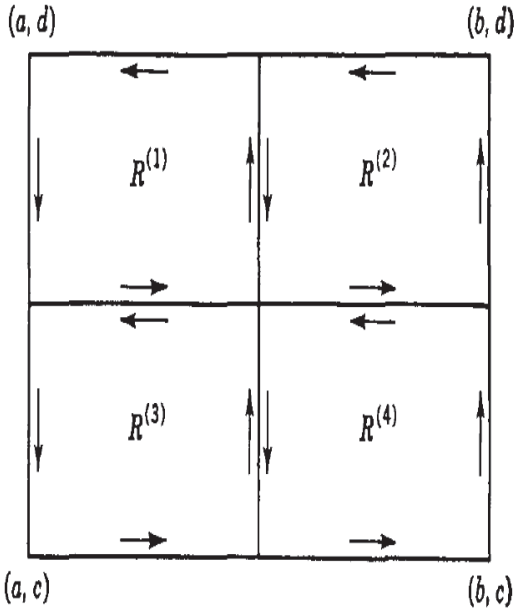
Corollary (Cauchy)

Let $f \in \mathcal{A}(B(z_0, r))$. Then for any closed $\gamma \subset B(z_0, r)$, $\oint_\gamma f(z)\,dz = 0$, and there exists $F \in \mathcal{A}(B(z_0, r))$ with $F'(z) = f(z)$ for all $z \in B(z_0, r)$.

Note that, unlike the "classical" Cauchy theorem, *no* continuity of f' is assumed here – this is the key improvement of Goursat's argument.

1 Proof of the Theorem

Key idea: bisection. Cut R into four equal rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$, oriented so that shared internal edges are traversed in opposite directions and cancel:



Observe:

$$\oint_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \oint_{\partial R^{(j)}} f(z)\,dz, \qquad \ell(\partial R^{(j)}) = \tfrac{1}{2}\ell(\partial R), \qquad \operatorname{diam}(R^{(j)}) = \tfrac{1}{2}\operatorname{diam} R.$$

Iterating. Suppose, towards a contradiction, $|\oint_{\partial R} f(z)\,dz| = A \neq 0$. Let $R_0 := R$. By the pigeonhole principle, some $R^{(j)}$ satisfies $|\oint_{\partial R^{(j)}} f(z)\,dz| \geq A/4$; let $R_1 := R^{(j)}$ for such j. Repeating,

$$\exists\, R_n \text{ a subrectangle of } R_{n-1}: \quad \left|\oint_{\partial R_n} f(z)\,dz\right| \geq A \cdot 4^{-n},$$
$$\ell(\partial R_n) = 2^{-n}\ell(\partial R), \qquad \operatorname{diam}(R_n) = 2^{-n}\operatorname{diam} R.$$

The limit point. Let $\{z^*\} = \bigcap_n R_n$ (nonempty and a single point, since the R_n are nested compact sets with $\operatorname{diam} R_n \leq 2^{-n}\operatorname{diam} R \to 0$). Pick $r > 0$ with $f \in \mathcal{A}(B(z^*, r))$ (possible since $z^* \in R$ and f is analytic on an open set containing R).

Since $f'(z^*)$ exists, for every $\varepsilon > 0$ there is $0 < \delta < r$ such that

$$|z - z^*| < \delta \implies |f(z) - f(z^*) - f'(z^*)(z - z^*)| < \varepsilon|z - z^*|.$$

Choose n with $2^{-n}\operatorname{diam}(R) < \delta$, so $\partial R_n \subset B(z^*, \delta)$. Split

$$\oint_{\partial R_n} f(z)\,dz = \underbrace{\oint_{\partial R_n} \Big[f(z) - f(z^*) - f'(z^*)(z - z^*)\Big]dz}_{\text{I}} + \underbrace{\oint_{\partial R_n} f(z^*)\,dz}_{\text{II}} + \underbrace{\oint_{\partial R_n} f'(z^*)(z - z^*)\,dz}_{\text{III}}.$$

Since $f(z^*) = \big(zf(z^*)\big)'$ and $f'(z^*)(z - z^*) = \Big(\frac{f'(z^*)}{2}(z - z^*)^2\Big)'$ are both exact (derivatives of single-valued functions), $\text{II} = \text{III} = 0$ over the closed loop ∂R_n. But

$$|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \varepsilon|z - z^*| \leq \varepsilon \operatorname{diam}(R_n) \qquad \text{for } z \in \partial R_n,$$

so

$$4^{-n}A \le \left|\oint_{\partial R_n} f(z)\,dz\right| = |\mathrm{I}| \le (\varepsilon \operatorname{diam}(R_n))\ell(\partial R_n) = 4^{-n}\,\varepsilon \operatorname{diam}(R)\,\ell(\partial R).$$

Dividing by 4^{-n}: $A \le \varepsilon \operatorname{diam}(R)\ell(\partial R)$ for *every* $\varepsilon > 0$. Assuming $A > 0$, take $\varepsilon = \dfrac{A}{2\operatorname{diam}(R)\ell(\partial R)}$: this gives $A \le A/2$, a contradiction. So $A = 0$. ■

2 The Improved Goursat Theorem

Theorem (Improved Goursat)

Let R be a rectangle, $R' := R\backslash\{z_1, \dots, z_n\}$ with $z_j \in \operatorname{Int}(R)$. Let $f \in \mathcal{A}(R')$ with $\lim\limits_{z\to z_j}(z-z_j)f(z) = 0$ for every j. Then $\oint_{\partial R} f(z)\,dz = 0$.

Remark. If f is continuous at z_j, or even just locally bounded near z_j ($\exists M, \delta : 0 < |z - z_j| < \delta \implies |f(z)| < M$), then automatically $\lim_{z\to z_j}(z - z_j)f(z) = 0$.

Corollary

Let $f \in \mathcal{A}(B(z_0, r) \setminus \{z_1, \dots, z_n\})$ with $\lim_{z\to z_j}(z - z_j)f(z) = 0$ for every j. Then for any closed $\gamma \subset B(z_0, r) \setminus \{z_1, \dots, z_n\}$, $\oint_\gamma f(z)\,dz = 0$, and there exists $F \in \mathcal{A}(B(z_0, r) \setminus \{z_1, \dots, z_n\})$ with $F' = f$ throughout.

Proof of the Improved Goursat Theorem. By subdividing R, we may assume $n = 1$; write z_1 for the single excluded point.

Cut R into 9 rectangles $R_1, \dots, R_9$ in a 3×3 grid, with z_1 inside the central rectangle R_5. For $j \neq 5$, $f \in \mathcal{A}(R_j)$, so $\oint_{\partial R_j} f(z)\,dz = 0$ by the ordinary Cauchy–Goursat theorem. Hence

$$\oint_{\partial R} f(z)\,dz = \sum_{j=1}^{9} \oint_{\partial R_j} f(z)\,dz = \oint_{\partial R_5} f(z)\,dz.$$

Fix $\varepsilon > 0$. Choose $\delta > 0$ such that $|z - z_1| < \delta \implies |f(z)(z - z_1)| < \varepsilon$, i.e. $|f(z)| < \varepsilon/|z - z_1|$. Take R_5 to be a square of side $\delta/2$ centered at z_1. Then for $z \in \partial R_5$,

$$\frac{\delta}{4} \le |z - z_1| \le \frac{\delta}{\sqrt{2}} < \delta, \qquad \text{so} \qquad |f(z)| < \frac{\varepsilon}{|z - z_1|} \le \frac{4\varepsilon}{\delta}.$$

So

$$\left|\oint_{\partial R} f(z)\,dz\right| = \left|\oint_{\partial R_5} f(z)\,dz\right| \le \varepsilon \oint_{\partial R_5} \frac{|dz|}{|z - z_1|} \le \varepsilon \cdot \frac{4}{\delta} \cdot \ell(\partial R_5) = \varepsilon \cdot \frac{4}{\delta} \cdot 2\delta = 8\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\oint_{\partial R} f(z)\,dz = 0$. ■